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1985 J. Phys. A: Math. Gen. 18 L513

(http://iopscience.iop.org/0305-4470/18/9/005)

J. Phys. A: Math. Gen. 18 (1985) L513-L515. Printed in Great Britain

LETTER TO THE EDITOR

Linear response and fluctuations in stochastic mechanics

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Received 29 March 1985

Abstract. The relation between response function and correlation function is derived for the stochastic process associated to the ground state within the framework of stochastic mechanics. The new relationship clarifies the energy-conserving character of Nelson's stochastic processes.

In Nelson stochastic mechanics (Nelson 1966, 1984a, b) quantum states are represented as classical Markov random processes obeying stochastic differential equations. Formally these equations are identical to the Langevin equation for the Brownian motion, the physics however is fundamentally different (Guerra 1981, Figari *et al* 1984). In the Langevin case the fluctuations are due to the environment and are intimately related to dissipative and time irreversible behaviour. Conversely, in Nelson stochastic mechanics fluctuations are intrinsic to the microscopic description and are therefore compatible with energy conservation and time reversible behaviour.

The dissipative character of the dynamics in the Langevin scheme induces the relation between response functions and correlation functions which goes under the name of fluctuation-dissipation theorem (FDT) (Callen and Welton 1951, Kubo 1957, 1966). It is then interesting, as a further contribution to the understanding of the stochastic processes arising in stochastic mechanics, to investigate the novel form of the relation between response and correlation functions which is expected within Nelson's dynamical scheme. In this case in fact such a relationship must reflect the energy conserving character of the stochastic time evolution.

Consider a particle in a potential $V(x)$ and subjected to a small time-dependent perturbation of the form $-\lambda x \cos \omega t$, where x is the position of the particle. The linear response function is real and it is given by (Landau and Lifshitz 1967)

$$\chi'(\omega) = \frac{2}{\hbar} \sum_m \frac{\omega_{m0} |x_{m0}|^2}{\omega_{m0}^2 - \omega^2} \tag{1}$$

where $\omega_{m0} = (E_m - E_0)/\hbar$, $x_{m0} = \langle m|x|0\rangle$ and $\omega \neq \omega_{m0}$.

Next, consider the position correlation function in the unperturbed equilibrium state (ground state). According to Nelson each wavefunction $\psi(x, t)$ is associated to a

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0305-4470/85/090513+03$02.25

Markov random process $x(t)$ which obeys the Itô stochastic differential equation:

$$dx = b(x, t)\, dt + dw \tag{2}$$

where $w(t)$ is a Wiener process with expectation values $\langle dw(t)\rangle = 0$, $\langle dw^2(t)\rangle = \hbar/m$ and the drift is related to the wavefunction $\psi = \exp(R + iS/\hbar)$ by

$$b(x, t) = \frac{1}{m}\frac{\partial}{\partial x}[S(x, t) + \hbar R(x, t)]. \tag{3}$$

For the ground-state wavefunction the drift is time independent and the above equation can be rewritten as

$$b(x, t) = -\partial W/\partial x \tag{4}$$

with $W(x)$ satisfying the Riccati equation

$$V - E_0 = \frac{m}{2\hbar}\left(\frac{\partial W}{\partial x}\right)^2 - \frac{1}{2}\frac{\partial^2 W}{\partial x^2}. \tag{5}$$

An equation of the form (5) arises also in a different context (see, for example, Van Kampen 1977), namely when the Fokker-Planck equation associated to the stochastic differential equation (2) is mapped into the imaginary time Schrödinger equation with the potential $V(x) - E_0$. Consequently, the transition probability of the Nelson stochastic process associated to the ground state can be written in the form

$$P(x't|x0) = \frac{\varphi_0(x')}{\varphi_0(x)}\sum_m \varphi_m^{(x)}\varphi_m^{(x')} \exp[-(E_m - E_0)t/\hbar] \tag{6}$$

where $\varphi_m(x)$ and E_m are the eigenfunctions and eigenvalues of the potential $V(x)$.

Assuming for simplicity that $V(x)$ is an even function of x, so that the ground state is $\langle x\rangle = 0$, we compute the position correlation function from

$$S_{xx}(t) = \langle x(0)x(t)\rangle = \int dx \int dx'\, \rho_0(x)P(x't|x0)xx' \tag{7}$$

where $\rho_0(x) = \varphi_0^2(x)$ is the ground-state probability density. Using equation (6) it is easy to obtain

$$S_{xx}(\omega) = \frac{2}{\hbar}\sum_m \frac{\omega_{m0}|x_{m0}|^2}{\omega_{m0}^2 + \omega^2} \tag{8}$$

where $S_{xx}(\omega)$ is the Fourier transform of the Nelson correlation function. Finally, comparing equations (1) and (8) we find

$$\chi'(i\omega) = S_{xx}(\omega)/\hbar \tag{9}$$

which yields the relation between fluctuations and response function in the Nelson process associated to the ground state. In order to establish the relationship between the above result and the formulation of the FDT in quantum mechanics, we recall some formal results of Green's function theory (Rickayzen 1984). Introducing the time-ordered correlation function $G(t) = \langle T[x(t)x(0)]\rangle$ and the symmetrised correlation function $C(t) = \frac{1}{2}\langle x(t)x(0) + x(0)x(t)\rangle$ where the average is taken over the thermal equilibrium state, the Fourier transforms of the above quantities are related to the real part $\chi'(\omega)$ of the response function by

$$G(\omega) = -i\hbar\chi'(\omega) + C(\omega). \tag{10}$$

Since $C(\omega)$ is related to the imaginary part $\chi''(\omega)$ of the response function by the FDT

$$C(\omega)=\hbar \coth(\beta\hbar\omega/2)\chi''(\omega) \tag{11}$$

in the ground state ($\beta\to\infty$) from equation (10) we find

$$G(\omega)=-\mathrm{i}\hbar\chi(\omega). \tag{12}$$

On the other hand, from causality, the response function is analytical in the upper half plane. Consequently, in the ground state, $G(\omega)$ is the limit value on the real axis of a function $G(z)$ analytic on the upper half plane and defined by $G(z)=-\mathrm{i}\hbar\chi(z)$. In particular on the imaginary axis, where the response function is real, one has

$$G(\mathrm{i}\omega)=-\mathrm{i}\hbar\chi'(\mathrm{i}\omega). \tag{13}$$

Finally, recalling (Guerra and Ruggiero 1973) that the time-ordered correlation function and the Nelson correlation function in the ground state are related by the Wick rotation $S(t)=G(-\mathrm{i}t)$, we find that equation (9) is equivalent to equation (13). In other words, the vanishing of the imaginary part of the response function along the imaginary axis indicates the existence of non-dissipative dynamics and the Nelson process provides the realisation, in the physical real time, of such time evolution.

One of us (PR) wishes to thank Professor Edward Nelson for many useful conversations and the Department of Mathematics, Princeton University for its kind hospitality.

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